Functional equation on $\mathbb{N}$.

https://www.linkedin.com/feed/update/urn:li:activity:6515582090648657920

Consider the (infinite) sequence of integers $a_1, a_2, a_3, \ldots$ all of which are greater than zero. If they satisfy the relation $a_n + a_{a_n} = 2n$ for all n, find a_n as a function of n.

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Let $f(n) := a_n$. Then by condition of the problem $f : \mathbb{N} \to \mathbb{N}$ and

$$f(n) + f(f(n)) = 2n, \ \forall n \in \mathbb{N}.$$

Since $f(1) \geq 1$ and $f(f(1)) \geq 1$ then $f(1) + f(f(1)) = 2 \Leftrightarrow f(1) = 2 - f(f(1))$ implies $f(1) \leq 1$. Therefore, $f(1) = 1$.

For any $n \in \mathbb{N}$ since $f(n) \geq 1$and $f(n) = 2n - f(f(n))$ we obtain $f(n) \leq 2n - 1$.

Hence, $f(f(n)) \leq 2f(n) - 1$ and then $f(n) = 2n - f(f(n)) \geq 2n - 2f(n) + 1 \Leftrightarrow$

$3f(n) \geq 2n + 1 \Leftrightarrow f(n) \geq \dfrac{2n+1}{3}, \forall n \in \mathbb{N}.$. Since $f(f(n)) \geq \dfrac{2f(n)+1}{3}$ then

$$f(n) = 2n - f(f(n)) \leq 2n - \frac{2f(n)+1}{3} \Leftrightarrow f(n) \leq \frac{6n-1}{5}.$$

Thus, $\dfrac{2n+1}{3} \leq f(n) \leq \dfrac{6n-1}{5} \Leftrightarrow \dfrac{2n+1}{3} - n \leq f(n) - n \leq \dfrac{6n-1}{5} - n \Leftrightarrow$

$$-\frac{n-1}{3} \leq f(n) - n \leq \frac{n-1}{5}$$

Let $a_1 = 3, b_1 = 5.$ Then assuming that for any $n \in \mathbb{N}$ holds inequality

$$-\frac{n-1}{a_k} \leq f(n) - n \leq \frac{n-1}{b_k} \ , \ k \in \mathbb{N}$$

we obtain $-\dfrac{n-1}{a_k} \leq f(n) - n \leq \dfrac{n-1}{b_k} \Rightarrow -\dfrac{f(n)-1}{a_k} \leq f(f(n)) - f(n) \leq \dfrac{f(n)-1}{b_k} \Leftrightarrow$

$$-\frac{f(n)-n+n-1}{a_k} \leq -2(f(n)-n) \leq \frac{f(n)-n+n-1}{b_k} \Leftrightarrow -\frac{n-1}{2b_k+1} \leq f(n) - n \leq \frac{n-1}{2a_k-1}$$

$a_{k+1} = 2b_k + 1, b_{k+1} = 2a_k - 1, k \in \mathbb{N}.$

$a_{k+1} + b_{k+1} = 2(a_k + b_k) \Rightarrow a_k + b_k = 2^{k-1}(a_1 + b_1) = 2^{k+2},$

$$a_{k+1} - b_{k+1} - \frac{2}{3} = -2\left(a_k - b_k - \frac{2}{3}\right) \Rightarrow a_k - b_k - \frac{2}{3} = (-2)^{k-1}\left(a_1 - b_1 - \frac{2}{3}\right) = \frac{(-2)^{k+2}}{3}$$

Hence $a_k = \dfrac{1}{2}\left(2^{k+2} + \dfrac{(-2)^{k+2}}{3}\right) = \dfrac{2^{k+1}\left(3 + (-1)^k\right)}{3},$

$$b_k = \frac{1}{2}\left(2^{k+2} - \frac{(-2)^{k+2}}{3}\right) = \frac{2^{k+1}\left(3 + (-1)^{k+1}\right)}{3}.$$

Since $\lim\limits_{k\to\infty}\left(-\dfrac{n-1}{a_k}\right) = \lim\limits_{k\to\infty}\dfrac{n-1}{b_k} = 0$ for any fixed $n \in \mathbb{N}$ then $f(n) = n, \forall n \in \mathbb{N}.$